6.2 Euler’s Method

Leonhard Euler (1707-1783)

Euler

Lagrange

Fourier

Dirichlet

Lipschitz

Klein

Story

Lefschetz

Tucker

Minsky

Winston

Waltz

Pollack

Levy

You

Motivation

- Accumulating a value (stock variable) requires us to integrate it
- Sometimes definite integral exists (analytical solution): $f(x) = x^2 \Rightarrow \int f(x)\, dx = x^3 / 3$
- Sometimes it does not: $f(x) = e^{-x^2}$
- In such cases we use numerical methods to approximate the integral (Euler, Runge-Kutta 2 and 4)

Example

- We use a somewhat artificial example where there *is* an analytical solution, to compare numerical methods with it.

- Consider $dP/dt = 0.10P$ with $P_0=100$. We know that $P = 100e^{0.10t}$

- General finite difference equation is

  $P(t) = P(t-\Delta t) + growth(t)*\Delta t$

- Let's compute $P(8)$ for this example

Example

- $dP/dt = 0.10P$ with P_0=100

- $P(t) = P(t\text{-}\Delta t) + growth(t)*\Delta t$ with $\Delta t = 8$

  $= P(t\text{-}\Delta t) + growth\_rate*P(t\text{-}\Delta t)*\Delta t$ with $\Delta t = 8$

  $= P(t\text{-}8) + 0.10 * P(t\text{-}8) * 8$

  $= P(t\text{-}8) + 0.80 * P(t\text{-}8)$

  $= P(t\text{-}8) * (1 + 0.80)$

  = 1.8*P(t-8)

- Let's compute P(8)….

Example

- $P(t) = 1.8*P(t-8)$

- $P(8) = 1.8*P(0)$

  $= 1.8*100$

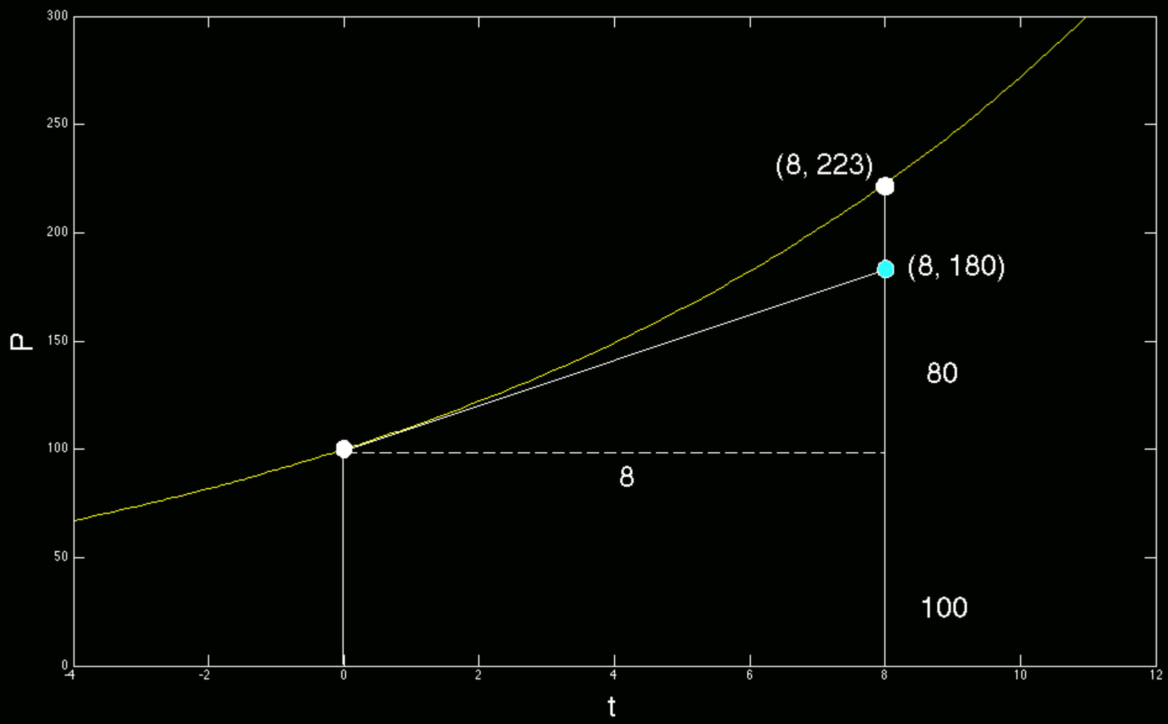
  $= 180$

- Check against analytical solution:

  $P = 100e^{0.10t}$

  $P_8 = 100e^{(0.10)(8)} = 223$

Example



Algorithm for Euler's Method

$t \leftarrow t_0$

$P(t_0) \leftarrow P_0$

Initialize *SimulationLength*

while t < *SimulationLength* do the following:

$t \leftarrow t + \Delta t$

$P(t) \leftarrow P(t - \Delta t) + P'(t - \Delta t) * \Delta t$

What's wrong with this picture?

Better Algorithm for Euler's Method

$t \leftarrow t_0$

$P(t_0) \leftarrow P_0$

Initialize *NumberOfSteps*

for *n* going from 1 to *NumberOfSteps* do the following:

$t_n \leftarrow t_0 + n\Delta t$

$P_n \leftarrow P_{n\text{-}1} + f(t_n\text{-}1, P_{n\text{-}1})\Delta t$

where *f* is the derivative at a given time step

Error Vanishes in the Limit

- For Δt = 8, relative error = |180-223|/|223 |= 19%
- For Δt = 1:

$$P(t) = P(t\text{-}1) + 0.10 * P(t\text{-}1) * 1$$
$$= P(t\text{-}1) * 1.1$$
$$= 100 * 1.1$$
$$= 110$$

$$P = 100e^{0.10t}$$
$$P_8 = 100e^{(0.10)(1)}$$
$$= 100e^{(0.10)(1)}$$
$$= 111$$

Relative error = |110-111|/|111| = 1%